RICCI-FLAT KÄHLER METRICS ON CANONICAL BUNDLES

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ABSTRACT. We prove the existence of a (unique) $S^1$-invariant Ricci-flat Kähler metric on a neighbourhood of the zero section in the canonical bundle of a real-analytic Kähler manifold $X$, extending the metric on $X$.

In the important paper [3], Calabi proved existence of Ricci-flat Kähler metrics on two classes of manifolds: a) cotangent bundles of projective spaces; b) canonical bundles of Kähler-Einstein manifolds. The metrics on $T^*\mathbb{CP}^n$ are actually hyperkähler and in the intervening years hyperkähler metrics were shown to exist on cotangent bundles of many other Kähler manifolds. Finally, recently, B. Feix [4] and, independently, D. Kaledin [9] have shown that a real-analytic Kähler metric on a complex manifold $X$ always extends to a (essentially unique) hyperkähler metric on a neighbourhood of $X$ in $T^*X$.

The aim of this paper is to prove the analogous generalization for the other class of Calabi’s metrics. Our main existence result can be stated as follows:

**Theorem 1.** Let $X$ be a real-analytic Kähler manifold. Then there exists a unique Ricci-flat Kähler metric on a neighbourhood of $X$ in the canonical bundle $K_X$ of $X$ which extends the metric on $X$ and for which the standard $S^1$-action on $K_X$ is isometric and Hamiltonian.

The condition of real-analyticity of the Kähler metric is clearly necessary, since the extended metric is Ricci-flat.

We also notice that the adjunction formula shows that the canonical bundle is the only line bundle over $X$ which can admit a Ricci-flat Kähler metric.

1. **Proof of Theorem 1**

Let $M$ be an $n+1$ dimensional Kähler manifold with a free Hamiltonian circle action. Then the metric can be locally written in the form:

$$G = \sum g_{ij} dz_i \otimes dz_j + wdt^2 + w^{-1} \phi^2,$$

where $t$ is the moment map on $M$, $\phi$ is the circle-invariant 1-form and the $z_i$ are local coordinates on $M/\mathbb{C}^\times$.

The complex structure $I$ maps $dt$ to $w^{-1}\phi$. Pedersen and Poon [11] (and LeBrun [10] for $n=1$) have worked out the conditions for the complex structure to be integrable and for the metric to be Einstein (in fact, Pedersen and Poon deal with the more general case of torus symmetry). We recall their theorem.

**Theorem 1.1.** [Pedersen-Poon] Let $w$ be a smooth positive function and $[g_{ij}]$ a positive definite hermitian matrix of smooth functions on an open set $U$ in $\mathbb{C}^i \times \mathbb{R}$.

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The metric (1.1) is Ricci-flat if and only if the following system of equations holds for some constant $c$:

$$4u_{z_i}z_j + c(g_{ij})_t = 0,$$

$$u_t = cw.$$  \hspace{1cm} (1.2)

$$4w_{z_i}z_j + (g_{ij})_t = 0.$$ \hspace{1cm} (1.4)

Here $u$ is defined by

$$\det g = we^u.$$ \hspace{1cm} (1.5)

Furthermore, the metric is defined on a circle bundle over $U$ if and only if the cohomology class $[F]$ of the curvature of $\phi$ which is given by

$$F = -\left( \frac{i}{2} (g_{ij})_t dz_i \wedge dz_j + i w_{z_i} dt \wedge dz_i - iw_{z_j} dt \wedge dz_j \right)$$ \hspace{1cm} (1.6)

belongs to $2\pi \mathbb{Z}$.

The constant $c$ has the following significance:

**Proposition 1.2.** $\Delta_G t = c$.

**Proof.** For any function $f$ we have

$$\Delta_G f = g^{ij} \left( \frac{4}{2} \frac{\partial f}{\partial z_i} \frac{\partial g_{ij}}{\partial z_j} + w^{-1} \frac{\partial f}{\partial t} \frac{\partial g_{ij}}{\partial t} \right).$$ \hspace{1cm} (1.7)

Thus, for $f = t$, we obtain

$$\Delta_G t = g^{ij} w^{-1} \frac{\partial g_{ij}}{\partial t} + \frac{\partial w^{-1}}{\partial t} = g^{ij} \frac{\partial w^{-1}}{\partial t} = w^{-1} \frac{\partial \ln \det g}{\partial t} + \frac{\partial w^{-1}}{\partial t}.$$ \hspace{1cm} (1.8)

Now, using (1.5) and (1.3), we have

$$\Delta_G t = w^{-1} \frac{\partial \ln \det g}{\partial t} + \frac{\partial w^{-1}}{\partial t} = w^{-1} \frac{\partial \ln w}{\partial t} + c + \frac{\partial w^{-1}}{\partial t} = c.$$ \hspace{1cm} \Box

For hyperkähler manifolds, the constant $c$ can take only two values:

**Proposition 1.3.** Let $M^{4n}$ be a hyperkähler manifold equipped with an isometric and Hamiltonian (for one symplectic structure) action of the circle. Then the moment map $t$ for this action is harmonic if the action is tri-holomorphic and satisfies $\Delta t = n$ otherwise.

**Proof.** If the action is triholomorphic, then the moment map is the real part of a complex moment map. If the action is not triholomorphic, i.e. it rotates the complex structures orthogonal to $I$, then the moment map is a Kähler potential for another Kähler form (corresponding to the complex structure $J$) and the result follows. \hspace{1cm} \Box

We shall seek metrics with $c \neq 0$. In this case the equation (1.4) is the consequence of the other two equations. Moreover we can eliminate the function $w$ from the equations and replace (1.3) and (1.5) with

$$(e^u)_t = c \det g.$$ \hspace{1cm} (1.8)
Suppose now that we are given a Kähler metric $h = \sum h_{ij} dz_i \otimes d\bar{z}_j$ on a complex $n$-dimensional manifold $X$ and we wish to extend $h$ to a Ricci-flat Kähler metric $g$ in a neighbourhood of $X$ in a line bundle $L$. Furthermore we require that the canonical $S^1$-action on $L$ is Hamiltonian. Clearly, a necessary condition for this is that we can solve the following (singular) Cauchy problem:

$$
\begin{align*}
&\begin{cases}
u_{x_i x_j} + u_{y_i y_j} + c(g_{ij})_t = 0 \\
(e^s)_t = c \det g \\
(g_{ij})_{t=0} = h_{ij} \\
(e^s)_{t=0} = 0
\end{cases} \quad (1.9)
\end{align*}
$$

Here $z_i = x_i + \sqrt{-1} y_i$ and the last condition is the consequence of the fact that $w^{-1} \equiv 0$ at $t = 0$ (since the circle acts trivially on $X$).

Our first result, which is a singular Cauchy-Kovalevskaya theorem, says that we can indeed solve this system locally, if the initial data $h_{ij}$ is real-analytic.

Theorem 1.4. Let $h_{ij}$, $i, j = 1, \ldots, n$, be real-analytic functions on an open subset $U$ of $\mathbb{C}^n$. Then there exists a unique solution of the system (1.9) on an open neighbourhood of $U$ in $U \times [0, +\infty)$.

Remark 1.5. Observe that, if the solution does give the metric on a neighbourhood of $X$ in $L$, then the initial data $h_{ij}$ is real-analytic, since the metric $g$ is Ricci-flat.

Proof. We treat $\mathbb{C}^n$ as a real subspace $V$ of $\mathbb{C}^{2n}$. Since $h_{ij}$ are real analytic, they extend to holomorphic functions on a neighbourhood of $V$. Therefore we can treat problem (1.9) as purely holomorphic, i.e. $x_i, y_j$ are complex coordinates. First of all, it is easy to see that (1.9) has a unique formal solution, i.e. a power series in $t$. Thus we only have to show that this series is convergent. Let us write $e^u = te^v$. Then we can rewrite the problem (1.9) as

$$
\begin{align*}
&\begin{cases}
u_{x_i x_j} + u_{y_i y_j} + c(g_{ij})_t = 0 \\
v_{x_i x_j} + v_{y_i y_j} + c(g_{ij})_t = 0
\end{cases} \quad (1.10)
\end{align*}
$$

with the initial conditions $(g_{ij})_{t=0} = h_{ij}$, $(e^s)_{t=0} = c \det g$. A theorem showing convergence of a formal solution to this system is proved in the appendix. This theorem is a slight generalization of a theorem of Gérard and Tahara [6]. It is applied to functions $\tilde{g}_{ij} = g_{ij} - h_{ij}$ and $\tilde{v} = v - v_0$, where $v_0 = v_{t=0}$.

Having solved the Cauchy problem (1.9) we ask whether the solution gives us a smooth metric (1.1) on a neighbourhood of $X$ in some line bundle $L$. First of all, we have

Lemma 1.6. Suppose that we have a local solution of the Cauchy problem (1.9) (with $c \neq 0$). Then the metric (1.1) extends smoothly to the hypersurface $t = 0$ (which is the fixed-point set of the circle action) if and only if $c = 1$.

Proof. Since $\det g$ is finite and non-zero at $t = 0$, equation (1.8) implies that $e^u = t(a + bt + \ldots)$ near $t = 0$ with $a \neq 0$. Therefore $u_t = \frac{1}{c} + O(1)$ near $t = 0$. Now $w = c^{-1} u_t$. This implies immediately that $c$ must be positive. Furthermore, the fibrewise metric is

$$
w(t)^2 + w^{-1} \phi^2 = \left( \frac{1}{ct} + O(1) \right) dt^2 + \left( \frac{1}{ct} + O(1) \right)^{-1} \phi^2.
$$
If we introduce a new coordinate $r$ so that $t = r^2$, we see that this metric extends smoothly to the origin if and only if $c = 1$. Now, the formula (1.6) shows that the connection 1-form $\phi$ and hence the metric (1.1) extends to the hypersurface $t = 0$. 

Thus it remains to show that the curvature form (1.6) belongs to $2\pi\mathbb{Z}$. We observe that the first equation in (1.9) (with $c = 1$) says that

$$
\frac{d}{dt} \omega = -i \partial \bar{\partial} u,
$$

(1.11)

where $\omega = \omega(t)$ is the Kähler form of the metric $g$ at time $t$ on $X$. On the other hand $u = \log (w^{-1} \det [g_{ij}])$. Since $g$ is Kähler (1.11) can be written as

$$
\frac{d}{dt} \omega = \rho(g) + i \partial \bar{\partial} \log w
$$

(1.12)

where $\rho$ denotes the Ricci form of a Kähler metric. Now (1.6) shows that the curvature form of $\phi$ is indeed in $2\pi\mathbb{Z}$, and in fact represents $-c_1(X)$. Thus Theorem 1 is proved.

2. Examples

We wish now to give explicit examples of Ricci-flat Kähler metrics on canonical bundles. Given a Kähler manifold $(X, h, I)$ we seek a time dependent metric $g = g(t)$ and a function $w$ on $X \times I$ which satisfy the equations (1.12), (1.3) and (1.5). The last two give us

$$
w^{-1} = \frac{\hbar \det g}{\det g},
$$

(2.1)

Substituting into (1.12) we obtain

$$
\frac{d}{dt} \omega = -i \partial \bar{\partial} \int_0^t \det g
$$

(2.2)

The first example deals with manifolds with constant principal Ricci curvatures, i.e. constant eigenvalues of the Ricci curvature. This class of manifolds includes both Kähler-Einstein manifolds and homogeneous manifolds. The following result is a particular case of a theorem of Hwang and Singer [8].

**Theorem 2.1.** Let $X^{2n}$ be a Kähler manifold with Kähler form $\Phi$ such that the eigenvalues of the Ricci curvature are constant. Then the solution to (2.2) is given by

$$
\omega = \Phi + t \rho(\Phi).
$$

(2.3)

The function $w^{-1}$ is given by

$$
w^{-1}(t) = \frac{\int_0^t P(t) dt}{P(t)},
$$

(2.4)

where $P(t)$ is defined as $P(t) = (\Phi + t \rho(\Phi))^n / \Phi^n$ (and so it depends only on the eigenvalues of the Ricci curvature).

In particular, if all the eigenvalues of the Ricci curvature are nonnegative, then the resulting Ricci-flat metric on $K_X$ is complete.

**Proof.** It is sufficient to observe that $\omega^n = P(t) \Phi^n$, so $\rho(\omega) = \rho(\Phi)$. Now it is clear that $\omega$ satisfies (2.2). 

$\square$
We remark that Apostolov, Armstrong and Dragici [1] recently found examples of irreducible non-homogeneous Kähler manifolds with constant principal Ricci curvatures.

As an aside, let us give an application to the geometry of Kähler quotients. We recall that Futaki [5] has shown that if \( M \) is a compact Kähler-Einstein manifold with positive scalar curvature and a Hamiltonian Killing vector field whose length is constant on the level sets of the moment map, then the Kähler quotient by the resulting circle action is also Kähler-Einstein. A simple example of \( \mathbb{C} \times \mathbb{C}^2 \) with the diagonal circle action on the second factor (and trivial on the first) shows that Futaki’s result does not hold for Ricci-flat manifolds. Nevertheless we have a weaker conclusion.

**Proposition 2.2.** Let \( M \) be a complete Ricci-flat Kähler manifold with an isometric and Hamiltonian circle action such that the length of the Killing vector field is constant on the level sets of the moment map. Moreover, assume that the moment map is bounded from below. Then the Kähler quotient of \( M \) by \( S^1 \) has constant principal Ricci curvatures.

**Proof.** Let \( t \) be the moment map and \( X = t^{-1}(a)/S^1 \) a particular Kähler quotient of \( M \). Since \( a \) is a regular value of \( t \), the Kähler quotients for nearby level sets are isomorphic to \( X \) (as a complex manifold). Thus we have a family \( g(t) \) of Kähler metrics on \( X \). The assumption and the equation (1.4) show that \( g(t) \) is linear in \( t \). Since it is bounded from below, Proposition 1.7 implies that the constant \( c \) of Theorem 1.1 is non-zero. Now the equations (1.3) and (1.5) show that \( \det g(t) = f(t) \det g(a) \) for \( t \) near \( a \). Therefore the Ricci form \( \rho(g) \) is constant in \( t \) and the equation (1.2) implies that \( \rho(g) \) is equal to \( \omega \), where \( \omega \) is the Kähler form of \( g \). The conclusion follows now, since we already know that \( \omega(t)^n = f(t)\omega(a)^n \).

The next examples involve surfaces of revolution. Let \( \Sigma \) denote either \( \mathbb{C} \) or \( \mathbb{C}P^1 \) with a metric of constant curvature. We define a surface \( \Sigma_a \) as the Kähler quotient of \( \Sigma \times \mathbb{C} \) by the action of \( \mathbb{R} \) defined as

\[
    r \times (x, z) = (e^{2\pi i a r} \cdot x, z + r).
\]

This is a surface of revolution and the Ricci-flat Kähler metric on \( K_{\Sigma_a} = T^*\Sigma_a \) is complete, since it can be obtained as a hyperkähler quotient of \( T^*\Sigma \times \mathbb{H} \) by \( \mathbb{R} \).

Now, the main result of [2] shows that these are all such surfaces of revolution:

**Proposition 2.3.** Let \( (X, h) \) be a surface of revolution such that the Ricci-flat Kähler metric defined in Theorem 1 is complete. Then \( (X, h) \) is isometric to one of the surfaces \( \Sigma_a \) defined above.

**Appendix A. A Cauchy-Kovalevskaya Theorem for a Class of Singular PDE’s**

In this section we shall prove a result about convergence of formal solutions to certain singular partial differential equations. This is a generalization of a theorem of Gérard and Tahara [6] to a class of singular systems of PDE’s and it uses their method of proof. The result of Gérard and Tahara applies to first order nonlinear PDE’s of the form:

\[
    g \left( t, x, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \right) = 0
\]
where \( g \) is a holomorphic function in some polydisc. As observed in [6], the convergence of formal solutions is no longer true if we allow \( g \) to depend on second derivatives of \( v \). We shall now show that the theorem remains valid for systems of PDE's of the following form:

\[
\begin{align*}
g(t, x, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n}, w_1, \ldots, w_N) &= 0 \\
\frac{\partial w_i}{\partial t} &= L_i(x)(v) + a_i(t, x), \quad i = 1, \ldots, N
\end{align*}
\]

where \( L_i(x) \) are linear differential operators of order at most 2. We remark that one can allow the dependence of \( L_i \) on \( t \), but this further complicates the already complicated notation.

To guarantee the existence of formal solutions we shall assume that the first equation can be written as:

\[
\left( t \frac{\partial}{\partial t} - \rho(x) \right) v = tb(x) + G(x) \left( t, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n}, w_1, \ldots, w_N \right)
\]

where \( \rho(x) \) and \( b(x) \) are holomorphic functions defined in a polydisc \( D \) centered at the origin of \( \mathbb{C}^n \), and

\[
G(x)(t, Z, V, X_i, Y_j, i \leq n, j \leq N) = \sum_{p+r+s+|\alpha|+|\beta| \geq 2} a_{p, r, s, \alpha, \beta}(x) t^p Z^r V^s X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_N^{\beta_n}.
\]

The coefficients \( a_{p, r, s, \alpha, \beta}(x) \) are holomorphic in \( D \) and

\[
|a_{p, r, s, \alpha, \beta}(x)| \leq A_{p, q, s, \alpha, \beta}.
\]

Moreover the power series

\[
\sum A_{p, q, s, \alpha, \beta} t^p Z^r U^s X^{\alpha} Y^{\beta}
\]

is convergent near the origin.

We seek a holomorphic solution \((v, w_i)\) to the above system satisfying

\[
v(0, x) = w_i(0, x) \equiv 0, \quad i = 1, \ldots, N.
\]

A formal solution is a power series solution of the form

\[
\sum_{m \geq 1} v_m(x) t^m
\]

whose coefficients are holomorphic in \( D \).

The particular form of the system (A.1) allows us to rewrite it as a single differential-integral equation:

\[
\left( t \frac{\partial}{\partial t} - \rho(x) \right) v = tb(x) + G(x) \left( t, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n}, \int_0^t L_1(v), \ldots, \int_0^t L_N(v) \right).
\]

Here we regrouped the terms in the power expansion of \( G \), so that \( G \) does not depend on \( \int_0^t a_i(t, x) \).

**Theorem A.1.** Each formal solution of (A.2) is convergent. If \( \rho(0) \not\in \mathbb{N}^n \), then there exists a unique formal solution satisfying \( v(0, x) \equiv 0 \).

**Proof.** If \( \rho(0) \not\in \mathbb{N}^n \), then (A.2) has a unique formal solution of the form

\[
\sum_{m \geq 1} v_m(x) t^m.
\]
Moreover, \( v_m(x) \) is determined recursively by the following formula:

\[
v_1(x) = \frac{b(x)}{1 - \rho(x)},
\]

and for \( m \geq 2 \)

\[
v_m(x) = \frac{1}{m - \rho(x)} f_m \left( v_1, 2v_2, \ldots, (m - 1)v_{m-1}, v_1, \ldots, v_{m-1}, \partial_i v_1, \ldots, \partial_n v_1, \ldots \right)
\]

\[
\partial_i v_{m-1}, \ldots, \partial_n v_{m-1}, \frac{1}{2} L_1(v_1), \ldots, \frac{1}{2} L_N(v_1), \frac{1}{m} L_1(v_{m-1}), \ldots, \frac{1}{m} L_N(v_{m-1});
\]

\[
\left\{ a_{p,q,s,\alpha,\beta} \right\}_{p+q+s+|\alpha|+|\beta| \leq m}.
\]

We shall show that this solution is convergent. Let \( D_\alpha \) denote the polydisc of diameter \( 2\alpha \). By taking \( R \) sufficiently small (in particular, \( R < 1 \)), we can assume that all the \( v_m(x) \) are holomorphic in \( D_R \) and we have:

\[
|v_1(x)| \leq A, \ |\partial_i v_1(x)| \leq A, \ i = 1, \ldots, n \ |L_j(v_1)(x)| \leq A, \ j = 1, \ldots, N;
\]

\[
|m - \rho(x)| \geq \sigma m, \ m = 1, 2, 3, \ldots.
\]

Moreover, let \( M \) be a constant such that the coefficients of

\[
L_i = \sum c_{i,k}(x) \frac{\partial^2}{\partial x_k \partial x_l} + \sum d_{i,k}(x) \frac{\partial}{\partial x_k} + e(x)
\]

satisfy

\[
\sum |c_{i,k}(x)| + \sum |d_{i,k}(x)| + |e(x)| \leq M.
\]

Now we consider the analytic equation:

\[
\sigma Y = \sigma A t + \frac{1}{(R - r)^2} \sum_{p+q+s+|\alpha|+|\beta| \geq 2} A_{p,q,s,\alpha,\beta} (2eY)^{(\sum \alpha)} (4e^2 MYt)^{(\sum \beta)}
\]

Here \( e \) is the smallest real number such that \( e^{\sqrt{-1}} = -1 \).

By the implicit function theorem, this equation has a unique analytic solution of the form

\[
Y = \sum_{m \geq 1} Y_m(r) t^m,
\]

determined by the following recursive formula

\[
Y_1 = A
\]

and, for \( m \geq 2 \),

\[
\sigma Y_m = \frac{1}{(R - r)^2} F_m \left( Y_1, \ldots, Y_{m-1}, 2eY_1, \ldots, 2eY_{m-1}, 4e^2 MY_1, \ldots, 4e^2 MY_{m-1}, \right)
\]

\[
\left\{ \left( \frac{A_{p,q,s,\alpha,\beta}}{(R - r)^{p+q+s+|\alpha|+|\beta| - 2}} \right)_{p+q+s+|\alpha|+|\beta| \leq m} \right\}.
\]

Moreover, by induction on \( m \), we see that \( Y_m(r) \) is expressed in the form

\[
Y_m(r) = \frac{C_m}{(R - r)^{2m-2}}, \ m = 1, 2, \ldots,
\]

with constants \( C_1 = A \) and \( C_m \geq 0 \) (for \( m \geq 2 \)).
We shall show that the power series for \( Y \) is a majorant power series for the formal solution (A.3). To do so, it is sufficient to prove the following inequalities for all \( m \):
\[
|v_m(x)| \leq m|v_m(x)| \leq Y_m(r) \text{ on } D_r \text{ for } 0 < r < R; \tag{A.7}
\]
\[
|\partial_i v_m(x)| \leq 2eY_m(r) \text{ on } D_r \text{ for } 0 < r < R, i = 1, \ldots, n; \tag{A.8}
\]
\[
|L_k (v_m)(x)| \leq 4e^2(m + 1)MY_m(r) \text{ on } D_r \text{ for } 0 < r < R, \quad k = 1, \ldots, N. \tag{A.9}
\]

The case \( m = 1 \) is clear from the definition of \( A \). We proceed by induction. We replace all the terms in (A.4) by their absolute values. Then we use the inductive assumption and also replace \( |a_{p,q,s,\alpha,\beta}| \) by \( \frac{A_{p,q,s+1,\alpha,\beta}}{(R-r)^{p+q+s+1}} \) (this is a majorant, as \( R < 1 \)). This has also the effect of replacing \( f_m \) by \( F_m \), and, from (A.5), it gives:
\[
|v_m(x)| \leq \frac{1}{m}(R - r)^2 Y_m(r) \tag{A.10}
\]
which proves (A.7) (cf. [6], p. 985). Since \( Y_m(r) \) has the form (A.6), the above inequality can be written as:
\[
|v_m(r)| \leq \frac{C_m}{m (R - r)^{2m-4}}.
\]
Now the following lemma proves (A.8) and (A.9).

**Lemma A.2.** If a function \( v(x) \) holomorphic in \( D_R \) satisfies
\[
|v(x)| \leq \frac{C}{(R - r)^p} \text{ on } D_r \text{ for } 0 < r < R,
\]
then
\[
|\partial_i v(x)| \leq \frac{Ce(p + 1)}{(R - r)^{p+1}} \text{ on } D_r \text{ for } 0 < r < R, \quad i = 1, \ldots, n.
\]

For the proof see [7], Lemma 5.1.3.

We now assume that \( \rho(0) = k \in \mathbb{N}^* \). We can modify \( \{A_{p,q,s,\alpha,\beta}\}_{p+q+s+1|\alpha|+|\beta| \leq k} \) so that \( v_k(x) \) satisfies (A.7)-(A.9) and then apply the previous proof.

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