BETTI NUMBERS OF 3-SASAKIAN QUOTIENTS OF SPHERES BY TORI

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ABSTRACT. We give a formula for the Betti numbers of 3-Sasakian manifolds or orbifolds which can be obtained as 3-Sasakian quotients of a sphere by a torus. This answers a question of Galicki and Salamon about the topology of 3-Sasakian manifolds.

A $(4m+3)$-dimensional manifold is 3-Sasakian if it possesses a Riemannian metric with three orthonormal Killing fields defining a local $SU(2)$-action and satisfying a curvature condition. A complete 3-Sasakian manifold $S$ is compact and its metric is Einstein with scalar curvature $2(2m+1)(2m+3)$. Moreover the local action extends to a global action of $SO(3)$ or $Sp(1)$ and the quotient of $S$ is a quaternionic Kähler orbifold.

A large family of compact non-homogeneous 3-Sasakian manifolds was found by Boyer, Galicki and Mann in [BGM2]. They are obtained by the 3-Sasakian reduction procedure, analogous to the symplectic or hyperkähler quotient construction, from the standard $(4m+3)$-sphere. Recently, in [BGMR], Boyer, Galicki, Mann and Rees have calculated the second Betti number of a 7-dimensional 3-Sasakian quotient of the $(4q+7)$-sphere by a torus, as being equal to $q$. Using the ideas from [BD], we shall give a formula for the Betti numbers of 3-Sasakian quotients of spheres by tori, valid in arbitrary dimension.

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**Theorem 1.** Let $S$ be a 3-Sasakian orbifold of dimension $4n - 1$ which can be obtained as a 3-Sasakian quotient of the standard $(4n + 4q - 1)$-sphere by a $q$-dimensional torus $N \leq Sp(n + q)$. Then the Betti numbers of $S$ depend only on $n$ and $q$ and are given by the following formula

$$b_{2k} = \dim H^{2k}(S, \mathbb{Q}) = \binom{q + k - 1}{k}$$

for $k \leq n - 1$.

**Remarks.** 1. Galicki and Salamon [GS] have shown that the odd Betti numbers $b_{2k+1}$ of any $(4n - 1)$-dimensional 3-Sasakian manifold vanish for $0 \leq k \leq n - 1$. Our proof reproduces this result for orbifolds satisfying the assumptions of Theorem 1. The Poincaré duality gives now the remaining Betti numbers $b_p$, $p \geq 2n$, of $S$.

2. The quotient of $S$ by any 1-PS of $SO(3)$ is a contact Fano orbifold $Z$. Theorem 1 in conjunction with Theorem 1.4 in [BG] gives the Betti numbers of $Z$.

3. For any $n > 2$, there is a bound on $q$ ($q \leq 2^n - n - 1$) in order for $S$ to be smooth (see Remark 1.3). There also is a total bound on both $n$ and $q$ [Ga].

4. The formula of Theorem 1 gives also the Betti numbers of “generic” toric hyperkähler orbifolds; see section 3.

Let us discuss some consequences of Theorem 1.

A compact 3-Sasakian manifold is *regular* if its quotient by the $SO(3)$ or $Sp(1)$ action is a (quaternionic Kähler) manifold. At present the only known regular 3-Sasakian manifolds of dimension greater than 3 are homogeneous and in $1 - 1$ correspondence with simple Lie algebras [BGM2].

Galicki and Salamon [GS] have shown that the Betti numbers of a regular 3-Sasakian manifold of dimension $4n - 1$ must satisfy the following relation:

$$(*) \quad \sum_{k=1}^{n-1} k(n-k)(n-2k)b_{2k} = 0.$$

These authors also asked if (*) holds for arbitrary 3-Sasakian manifolds. Theorem 1 shows that this relation is intimately related to $S$ being regular:
Proposition 2. Let $S$ be a $3$-Sasakian manifold satisfying the assumptions of Theorem 1 with $n \geq 3$. Then the Betti numbers of $S$ satisfy the relation (*) if and only if $q = 1$, i.e. $S$ has Betti numbers of the homogeneous $3$-Sasakian manifold of type $A_n$.

Remark. There are smooth quotients with $q > 1$ - see Theorem 4.1 in [BD] (reproduced as Theorem 1.2 below) or Theorem 2.14 in [BGR]. For example, if $n = 3$, then $q$ can be $1, 2, 3$ or $4$.

Corollary 3. Let $S$ be a $3$-Sasakian manifold satisfying the assumptions of Theorem 1 with $n > 1$. Then $S$ is regular if and only if $S$ is homogeneous.

1. 3-SASAKIAN AND HYPERKÄHLER QUOTIENTS BY TORI

A $4n$-dimensional manifold is hyperkähler if it possesses a Riemannian metric $g$ which is Kähler with respect to three complex structures $J_1, J_2, J_3$ satisfying the quaternionic relations $J_1 J_2 = -J_2 J_1 = J_3$ etc. Such a manifold is automatically Ricci flat.

Instead of giving the intrinsic definition of a $3$-Sasakian manifold, which can be found in [Bä,BGM1-2,GS], we simply recall that a Riemannian manifold $(S, g)$ is $3$-Sasakian if and only if the Riemannian cone $C(S) = (\mathbb{R}^+ \times S, dr^2 + r^2 g)$ is hyperkähler [Bä,BGM2]. The three Killing vector fields on $S$, defining the local $Sp(1)$ action, are then given by $\xi_i = J_i \frac{\partial}{\partial r}$ (we identify $S$ with $S \times \{1\} \subset C(S)$).

We shall now quickly review the hyperkähler and $3$-Sasakian quotients by tori (see [BD] for more information). We consider the diagonal maximal torus $T^d$ of the standard representation of $Sp(d)$ on $\mathbb{H}^d$. A rational subtorus $N$ of $T^d$ is determined by a collection of nonzero integer vectors $\{u_1, \ldots, u_d\}$ (which we shall always take to be primitive) generating $\mathbb{R}^n$. For then we obtain exact sequences of vector spaces

\begin{equation}
0 \longrightarrow n \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \longrightarrow 0, \tag{1.1}
\end{equation}

\begin{equation}
0 \longrightarrow \mathbb{R}^n \xrightarrow{\beta^*} \mathbb{R}^d \xrightarrow{\iota^*} n^* \longrightarrow 0, \tag{1.2}
\end{equation}
where the map $\beta$ sends $e_i$ to $u_i$. There is a corresponding exact sequence of groups
\begin{equation}
1 \to N \to T^d \to T^n \to 1.
\end{equation}

The hyperkähler moment map $\mu = (\mu_1, \mu_2, \mu_3)$ for the action of $N$ is given by
\begin{equation}
\mu_1(z, w) = \frac{1}{2} \sum_{k=1}^{d} \left( |z_k|^2 - |w_k|^2 \right) \alpha_k + c_1
\end{equation}

\begin{equation}
(\mu_2 + \sqrt{-1} \mu_3)(z, w) = \sum_{k=1}^{d} (z_k w_k) \alpha_k + c_2 + \sqrt{-1} c_3.
\end{equation}

The constants $c_j$ are of the form
\begin{equation}
c_j = \sum_{k=1}^{d} \lambda_k^j \alpha_k, \quad (j = 1, 2, 3).
\end{equation}

where $\lambda_k^j \in \mathbb{R}$.

For our purposes it is enough to consider the case when $\lambda_k^2 = \lambda_k^3 = 0$ for $k = 1, \ldots, d$. We then write $\lambda_k = \lambda_k^1$, $k = 1, \ldots, d$, and we denote the hyperkähler quotient $\mu^{-1}(0)/N$ by $M(\underline{u}, \underline{\lambda})$ or sometimes just $M$.

In [BD] necessary and sufficient conditions for $M(\underline{u}, \underline{\lambda})$ to be a manifold or an orbifold were given. We shall only need the ones for an orbifold:

**Theorem 1.1 [BD].** Suppose we are given primitive integer vectors $u_1, \ldots, u_d$ generating $\mathbb{R}^n$ and real scalars $\lambda_1, \ldots, \lambda_d$ such that the hyperplanes $H_k = \{ y \in \mathbb{R}^n; \langle y, u_k \rangle = \lambda_k \}$, $k = 1, \ldots, d$, are distinct. Then the hyperkähler quotient $M(\underline{u}, \underline{\lambda})$ is an orbifold if and only if every $n + 1$ hyperplanes among the $H_k$ have empty intersection. \(\Box\)

If the condition of this theorem is satisfied we refer to $M = M(\underline{u}, \underline{\lambda})$ as a *toric hyperkähler orbifold*.

If we set all $\lambda_k$ equal to $0$, then the hyperkähler quotient or $M(\underline{u}, \underline{0})$ is the Riemannian cone over a (usually singular) 3-Sasakian space $S$. Equivalently $S$ is the 3-Sasakian quotient [BGM2] of the unit sphere in $\mathbb{H}^d$ by the torus $N$ (i.e. the quotient by $N$ of the 0-set of the restriction to the sphere of the hyperkähler moment map). We have (see also [BGMR])
Theorem 1.2 [BD]. Let \( u = (u_1, \ldots, u_d) \in \mathbb{Z}^d \) be a primitive collection of vectors generating \( \mathbb{R}^n \) and let \( N \) denote the corresponding torus defined by (1.4). Then the 3-Sasakian quotient \( S \) of the unit \((4d - 1)\)-sphere by \( N \) is manifold if and only if the following two conditions hold:

(i) every subset of \( u \) with \( n \) elements is linearly independent;

(ii) every subset of \( u \) with \( n - 1 \) elements is a part of a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^n \).

Condition (i) is necessary and sufficient for \( S \) to be an orbifold. \( \square \)

Remark 1.3. For \( n = 2 \), the vectors \( u_1, \ldots, u_d \) satisfy both conditions if each of them has relatively prime coordinates and each pair of vectors \( u_k \) is linearly independent.

On the other hand, if \( n \geq 3 \) and the vectors \( u_1, \ldots, u_d \) satisfy both conditions, then \( d < 2^n \). I am grateful to Krzysztof Galicki for informing me that Charles Boyer has found such a bound for \( n = 3 \) and to Gerd Mersmann for the following argument. Suppose there are \( 2^d \) such vectors. Then either a vector \( u_i \) has all coordinates equal to zero \( \mod 2 \) or two vectors \( u_i, u_j \) are equal \( \mod 2 \). In either case we obtain a subset \( \{u_i\} \) or \( \{u_i, u_j\} \) which cannot be a part of a \( \mathbb{Z} \)-basis.

Finally we shall need some facts from [BD] about the topology of a toric hyperkähler orbifold \( M = M(u, \Delta) \). The hyperplanes \( H_k \) of Theorem 1.1 divide \( \mathbb{R}^d \) into a finite family of closed convex polyhedra, some unbounded. We consider the polytopal complex \( C \) consisting of all bounded faces of these polyhedra. The support \( |C| \) of \( C \) is the union of all polyhedra in \( C \). If \( \phi = (\phi_1, \phi_2, \phi_3) : M \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) is the induced moment map for the action of \( T^n = T^d/N \) on \( M \), then it is shown in [BD] that the compact variety

\[
X = \phi^{-1}(|C|, 0, 0)
\]

is a deformation retract of \( M \). The variety \( X \) is a union of toric varieties and the usual formula for the Betti numbers of a toric variety (cf. [Fu]) holds for \( X \):

Theorem 1.4 [BD]. Let \( M = M(u, \Delta) \) be a toric hyperkähler orbifold. Then \( H^j(M, \mathbb{Q}) = 0 \) if \( j \) is odd and

\[
b_{2k} = \dim H^{2k}(M, \mathbb{Q}) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} d_i,
\]
where $d_i$ denotes the number of $i$-dimensional elements of the complex $C$.

2. Proof of Theorem 1

Let $d = n + q$. The idea is to consider a toric hyperkahler orbifold $M = M(u, \lambda)$ where the vectors $u_1, \ldots, u_d$ are the ones defining the torus $N$ and to show that the infinity of $M$ is homeomorphic to $S$. Observe that the condition of Theorem 1.1 is satisfied for generic choice of scalars $\lambda_k$ if the vectors $u_k$ satisfy the condition (i) of Theorem 1.1. We shall show that $M \cup S$ is a certain quotient of the closed unit ball $\tilde{B}$ in $\mathbb{H}^d$.

Let $s$ be a diffeomorphism between $[0, 1]$ and $[0, +\infty]$ with $s'(0) = 1$, and let $f(r) = s(r)/r$. We define a “moment map” $\nu : \tilde{B} \to \mathfrak{n}^*$ by the formula

$$
\nu(q) = \frac{1}{f^2(||q||)} \mu(f(||q||)q)
$$

(2.1)

where $\mu$ is given by (1.5). We observe that

$$
\nu(q) = \mu(q) + \frac{1}{f^2(||q||)} c
$$

(2.2)

where $c$ is given by (1.5c). In particular, restricted to the unit sphere, $\nu$ is just the 3-Sasakian moment map. We denote by $\Sigma$ the 0-set of $\nu$ and by $\Sigma^0$ the intersection of $\Sigma$ with the open unit ball $B \subset \tilde{B}$.

We observe that $\Sigma$ is $T^d$-invariant and that $\Sigma^0$ is $T^d$-equivariantly homeomorphic to the 0-set of $\mu$. Therefore the quotient $\Sigma^0/N$ is $T^n$-equivariantly homeomorphic to $M = M(u, \lambda)$ and the compact Hausdorff space $\Sigma/N$ can be identified with $M \cup S$. Moreover, it follows from the proof of Theorem 6.5 in [BD] that the deformation $h : M \times [0, 1] \to M$, $h(m, 1) = m$, $h(M, 0) = X$, where $X$ is given by (1.5), extends to $S$ (it is important here that every $n$ among the vectors $u_k$ are independent, and, therefore, each of the unbounded $n$-dimensional polytopes in the complement of the hyperplanes $H_k$ of Theorem 1.1 has an $(n - 1)$-dimensional face at infinity). Therefore $\tilde{M} = \Sigma/N$ is homotopy equivalent to $X$.

We have the long exact sequence of rational cohomology

$$
\ldots \to H^k_c(M) \to H^k_c(\tilde{M}) \to H^k(S) \to H^{k+1}_c(M) \to \ldots
$$
Since $M$ is an orbifold, and so a rational homology manifold, we can apply Poincaré duality to $M$ and obtain $H^k(M) \approx H_{4n-k}(M) \simeq H_{4n-k}(X)$. If $k < 2n$, then $H_{4n-k}(X) = 0$ and so $H^k(S) \approx H^k(M) \simeq H^k(X)$ for $k < 2n - 1$.

It remains to calculate the number $d_i$ of $i$-dimensional elements of the complex $C$ and to apply the formula (1.6). As noticed above, since every $n$ among the vectors $u_k$ are independent, the number $d_i$ depends only on $d$ and $n$. We use the formula 18.1.3 in [Gr] giving the number $f_i(d, n)$ of $i$-dimensional faces of the simple (i.e. no more than $n$ of the hyperplanes have a nonempty intersection) arrangement $A$ of $d$ hyperplanes in $\mathbb{R}^p$:

$$f_i(d, n) = \binom{d}{n-i} \sum_{j=0}^i \binom{d-n-1+i}{j}.$$

The number of $i$-dimensional faces of the complex $C$ is the number of $i$-dimensional faces in the arrangement $A$ which do not meet the infinity in $\mathbb{R}^p$. In other words

$$d_i = f_i(d, n) - f_{i-1}(d, n-1) = \binom{d}{n-i} \binom{d-n-1+i}{i}.$$

This yields

$$\dim H^{2k}(S, \mathbb{Q}) = \sum_{i=k}^n (-1)^{i-k} \binom{i}{k} \binom{q+n}{n-i} \binom{q+i-1}{i}$$

for $k \leq n - 1$. We now use the simple identity

$$\binom{i}{k} \binom{q+i-1}{i} = \binom{q+i-1}{i-k} \binom{q+k-1}{k}$$

to rewrite the formula (2.3) as

$$\dim H^{2k}(S, \mathbb{Q}) = \binom{q+k-1}{k} \sum_{i=k}^n (-1)^{i-k} \binom{q+n}{n-i} \binom{q+i-1}{i-k}.$$
We observe that \( F(q,n,k,i) = F(q - 1, n + 1, k + 1, i + 1) \) and therefore \( G(q, n, k) = G(q - 1, n + 1, k + 1) \). It follows that \( G(q, n, k) = G(1, n + q - 1, k + q - 1) \). However for \( q = 1 \) the right-hand side of (2.4) must be \( 1 \) for \( k \leq n - 1 \), since the left-hand side is \( 1 \) for the homogeneous 3-Sasakian manifold of type \( A_n \) [GS]. Therefore \( G(q, n, k) = G(1, n + q - 1, k + q - 1) = 1 \) for all \( q \) and \( k \leq n - 1 \). This proves Theorem 1.

3. Consequences

We shall now prove Proposition 2 and Corollary 3. The formula (*) is invariant under the symmetry \( k \mapsto n - k \) and we can write it as

\[
\sum_{k=1}^{[(n-1)/2]} k(n - k)(n - 2k)(b_{2k} - b_{2(n-k)}) = 0.
\]

To prove Proposition 2 it is enough to show that, for \( q > 1 \), \( b_{2k} - b_{2(n-k)} < 0 \) for all \( 1 \leq k \leq [(n - 1)/2] \). By Theorem 1 this is equivalent to \( \frac{(q+n-k-1)!}{k!} > \frac{(q+k-1)!}{(n-k)!} \) for \( 1 \leq k \leq [(n - 1)/2] \). We can write both expressions as products of \( q - 1 \) terms such that each term on the left is greater than the respective term on the right. Proposition 2 follows. For Corollary 3 we observe that Proposition 2 implies that if \( n \geq 3 \), then \( q = 1 \), and so \( S \) is the 3-Sasakian quotient of a sphere by a circle. These were analyzed in detail by Boyer, Galicki and Mann in [BGM2] and the result follows in this case from their work. For \( n = 2 \) it is well-known that the only compact 4-dimensional self-dual Einstein manifolds are \( S^4 \) and \( \mathbb{C}P^2 \) [Hi]. This proves Corollary 3 in this case.

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